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MODELLING THE SINGULARITIES IN THE SOLUTION OF PROBLEMS IN THE THEORY OF ELASTICITY†

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Singularities in the solution of certain problems in the theory of elasticity of the contact problem type or problems concerning cracks) are analytically modelled in a formulation of variational problems for boundary functionals using a singular solution of the Lamé equation. Numerical modelling makes use of an uncoupled variational formulation of the boundary-element method.

Unlike the case when a singularity in the solution of a boundary-value problem follows naturally from its analytic solution, as occurs in the solutions of planar problems in the theory of elasticity using the apparatus of the theory of functions of a complex variable (examples are the quite extensive class of problems with singularities of the type of a cavity, notch, cusp, etc. [1, 2], contact problems for a punch with a rectangular base [3, 4] and problems on cracks [4]), the modelling of a singularity in a solution involves the construction of a model in which, to realize the singularity, a certain analytic or numerical procedure is incorporated (for example, the solution in series is supplemented with an expansion in a series for a δ-function which is concentrated at the singular point or, in the case of a finite-element approximation of the solution, "singular" elements are made use of in the neighbourhood of the singular point). There are a large number of recent publications dealing with the solution of problems in the theory of elasticity with singularities which make use of the apparatus of boundary integral equations and numerical approximations of the type of finite-element approximations, the apparatus of boundary integral equations and asymptotic expansions in the neighbourhood of singular points, etc.

In this paper, it is proposed that a numerical-analytic algorithm be used to model a singularity in a solution on the boundary of certain (planar and spatial) problems in the theory of elasticity. This algorithm leads to the implementation of the proposed [5–8] variational method of boundary elements (VMBE) in an uncoupled formulation [9]: the displacements and stresses are disconnected by way of certain relationships on the boundary which enable one to adopt an independent boundary-element approximation for them (of a higher order for the stresses in the neighbourhood of the singular point). Here, these characteristic relationships are satisfied as coupling equations, using Lagrange multipliers in the procedure for solving the double variational problem.

1. The VMBE algorithm [8] reduces a boundary-value problem on a boundary to the form of the equivalent problem of minimizing a boundary functional in the solutions of a homogeneous Lamé equation at the points of a domain. If the initial boundary-value problem is irregular in the sense that the boundary has a singular point (the tip of a crack, for example),

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then Betti's formula [10] (or Green's formula, if one is dealing with an abstract second-order elliptic boundary-value problem) is required which is used in reducing the problem to the boundary. With reference to the analysis carried out in [11, p. 23] for domains with a boundary which is continuous in the Lipshits sense, it may be asserted that, if an outward normal to the boundary exists "almost everywhere" (with the exception of a finite boundary set of measure zero, that is, the singular point or line) and the trace operator $\gamma: W_2^1(G) \ni \varphi \to \gamma\varphi \in W_2^{1/2}(S)$ also exists "almost everywhere", then Betti's formula holds and a formulation of a variational problem for the boundary functional is possible.

It is proposed that a singular solution of the Lamé equation should be used for the analytic modelling of a singularity. The idea lies in the fact that the result of the presence of a singular point is to be considered as a stress field which is generated by the action of a unit force applied at the singular point. Then, the normal component $\mathbf{g}^{(\nu)}(y)$ of the specified stresses at the boundary points $y \in S$ (on a crack contour, for example) will be represented as a sum of a regular component $\mathbf{g}_r^{(\nu)}$ of the stresses specified on the boundary (on a crack contour) $\boldsymbol{\sigma}^{(ij)}(y)$ $(i, j = 1, \ldots, m)$ and the singular component $\mathbf{g}_s^{(\nu)} \equiv \mathbf{T}^{(ij)}(y, y_0)$ $(y_0 \in S$ is the singular point of the boundary) which, in essence, are the singular kernels of the vector potential of a double layer [12, p. 219]. Hence, the singularity of a function of the stresses specified on a boundary

$$\mathbf{g}^{(\nu)} = \mathbf{g}_r^{(\nu)}(\boldsymbol{\sigma}^{(ij)}) + \mathbf{T}^{(ij)} \tag{1.1}$$

is responsible for the irregularity of the variational problem for the boundary functional

$$\min_{\mathbf{u} \in D} F(\mathbf{u}), \quad F(\mathbf{u}) = \int_S \mathbf{u}\mathbf{t}^{(\nu)}(\mathbf{u})ds - 2\int_S \mathbf{u}\mathbf{g}^{(\nu)}ds \tag{1.2}$$

$$D(F) = \{\mathbf{u}:\ \mathbf{A}\mathbf{u}(x) = 0,\ x \in G\}$$

where D is the set of permissible displacement vectors, $\mathbf{A}$ is the vector operator of the theory of elasticity and $\mathbf{t}^{(\nu)}(\mathbf{u})$ is the vector of the required stresses at points on the boundary S which bounds the domain G.

Because of the irregularity of problem (1.2), the question arises as to the existence of a solution. It is sufficient to establish the boundedness of the linear functional $l(\mathbf{u}) = \int_S \mathbf{u}\mathbf{T}^{(ij)}ds$ as a singular integral with a density $\mathbf{u}(y) \in L_2(S)$, $y \in S$ and with a singular kernel $\mathbf{T}^{(ij)}(y, y_0)$, y, $y_0 \in S$ which has a singularity $O(r_0^{-2})$, $r_0 = |y - y_0|$. We know [12] that, subject to certain conditions on the characteristic of the above-mentioned integral (which have no effect on the formulation of the problem under consideration), the singular integral operator which is generated by it is bounded in $L_2(S)$ [12, p. 127]. The existence of a solution of problem (1.2) then follows from the fact that a second problem which is equivalent to it [8] has a solution [10].

In connection with the question under discussion regarding the existence of a solution of the variational problem (1.2), which is associated with the problem of proving the convergence of the solution of the approximating (discrete) variational problem, it should be noted that the question of the existence of the discrete variational problem itself is of great significance. For example, in the case of a linear boundary element approximation of the crack contour (circular or elliptic in plan view), we obtain a polygonal boundary which is discontinuous in the Lipshits sense [11, p. 95]; the concept of multiple nodes [13, p. 196] enables one to "isolate" the singular point, that is, the crack tip and, under the conditions of conformal finite-element methods [11] (the boundary-element matching conditions are satisfied such that the global interpolation function which approximates the solution is continuous at all points of the discrete boundary), a solution of the corresponding discrete variational problem always exists [11, p. 26]. In this case, the grid point values at the multiple grid points of the singular functions $\mathbf{T}^{(ij)}$ which simulate the stress field in the neighbourhood of the tip of a crack are finite in the case of the formation of the discrete functional $F_\Delta(\mathbf{u}_N)$ (see below).

If $\mathbf{u}_0$ is a solution of (1.2), then u_0 satisfies the boundary variational equation

$$\int_S \mathbf{u}\mathbf{t}^{(\nu)}(\mathbf{u}_0)ds - \int_S \mathbf{u}\mathbf{g}^{(\nu)}ds = 0, \quad \forall \mathbf{u} \in D(F) \tag{1.3}$$

The VMBE algorithm reduces to an approximation and the solution of Eq. (1.3).

2. As has already been mentioned, numerical modelling of a singularity (the stress field in the neighbourhood of the crack tip, for example) makes use of an uncoupled formulation of VMBE [9] which must be modified due to the specific features of the solutions of problems in the theory of elasticity with singularities: the displacement field is regular in the neighbourhood of a singular point (the crack tip) while the stress field is singular (for example, Griffiths solution, etc.), whereupon the governing relationships are not satisfied locally. In such a case it is natural to employ a locally uncoupled formulation of VMBE in the approximation of these solutions: the governing relationships are satisfied as coupling equations in the variational problem on parts of the boundary (in the neighbourhood of the singular point) while coupled approximations are used on the remaining boundary. The algorithm is correspondingly modified [9], and the variational principle remains true: regardless of the fact whether the coupling equations are specified on the whole of the boundary or on parts of it, in a state of equilibrium the energy of the surface stresses has one and the same value for the corresponding displacements. What has been said is expressed by the duality principle which holds in the case of a modified Lagrangian (see [9, formula 1.4])

$$L_0(\mathbf{u}, \mathbf{t}^{(\nu)}, \boldsymbol{\lambda}) = F(\mathbf{u}) - \int_{S_0} \boldsymbol{\lambda}[\mathbf{t}^{(\nu)} - \mathbf{t}^{(\nu)}(\mathbf{u})]ds \tag{2.1}$$

Here $F(\mathbf{u})$ is the functional from (1.2) and the second term is the integral expression of the governing relationships in the case of the surface stress vector $\mathbf{t}^{(\nu)}$ on a segment of the boundary S_0 in the neighbourhood of the singular point. Here, the Lagrange multipliers λ have the meaning of displacements [9] and are subsequently identified with them. The system of variational equations to the solution of which the solution of the dual problem for L_0 leads [9], is written in the form

$$2\int_S \mathbf{v}\mathbf{t}^{(\nu)}(\mathbf{u})ds - 2\int_S \mathbf{v}\mathbf{g}^{(\nu)}ds + \int_{S_0} \boldsymbol{\lambda}\mathbf{t}^{(\nu)}(\mathbf{v})ds = 0 \tag{2.2}$$

$$\int_{S_0} \boldsymbol{\mu}[\mathbf{t}^{(\nu)} - \mathbf{t}^{(\nu)}(\mathbf{u})]ds = 0 \tag{2.3}$$

$\forall \mathbf{v}$, $\boldsymbol{\mu} \in D(F)$ (see (1.2)), and, here, the second of them corresponds to the coupling equation.

First, without using the approximation of boundary-element methods, we will analyse the solutions of Eqs (2.2) and (2.3) with approximations of the Bubnov–Galerkin type (the VMBE approximations proposed in [9] are in essence of this type). Let there be the following systems of coordinate functions: a complete system of vector functions $\{\boldsymbol{\varphi}_i\}$, that is, of homogeneous solutions of the equations of the theory of elasticity and a system of sufficiently smooth vector functions $\{\boldsymbol{\psi}_j\}$ defined at the point S_0. Let us consider the approximations for the required displacement and stress vectors

$$\mathbf{u}_k = \sum_{i=1}^{k} U_i\boldsymbol{\varphi}_i, \quad \mathbf{t}_n^{(\nu)} = \sum_{j=1}^{n} T_j\boldsymbol{\psi}_j \tag{2.4}$$

where U_i, T_j are the required coefficients. In Eq. (2.2), we identify $\boldsymbol{\lambda}$ and $\mathbf{u}$ and $\boldsymbol{\mu}$ with $\mathbf{v}$ in (2.3). The equations will obviously be satisfied for all vector functions $\mathbf{v}$ of the form $\mathbf{v}_k = \Sigma V_i\boldsymbol{\varphi}_i$ $(i = 1, \ldots, k)$, where V_i are arbitrary, and, then, from (2.2) and (2.3) using an expansion of the form $\mathbf{g}_k^{(\nu)} = \Sigma Q_i\boldsymbol{\varphi}_i$ $(i = 1, \ldots, k)$, for the required vector $\mathbf{g}^{(\nu)}$, we obtain the system of resolving equations in U_i and T_j.

Obtaining a numerical solution is associated with the question of the solvability of the subsystem of equations in U_i which is obtained from Eq. (2.2). The coefficients T_j are determined after the U_i have been found from the subsystem obtained from Eq. (2.3) (this follows from the fact that the unknown $\mathbf{t}^{(\nu)}$ only occur in the second equation of system (2.2), (2.3)).

The non-singular character of the matrix of the system holds if the spectrum of the matrix does not contain a zero eigenvalue.

In order to prove this, we employ spectral analysis of the integro-differential forms which appear in Eq. (2.2) (when $\boldsymbol{\lambda} \equiv \mathbf{u}$). We reduce the equation to an operator form using the apparatus of Sobolev spaces (with a fractional index) of vector functions defined at points of S (S is taken as being sufficiently smooth) and assuming that the coupling equations are specified on the whole of the boundary S, that is, $S_0 \equiv S$. We then return to the case being considered when $S_0 \subset S$. The above-mentioned spaces $W_2^1(G)$, $W_2^{1/2}(S)$ are spaces of the tracks on S of solutions from the Sobolev class $W_2^1(G)$ of variational problems of the type of (1.2). We shall denote scalar products in these spaces by $(.,.)_{1/2,S}$ and $(.,.)_{-1/2,S}$ respectively, and $\langle .,. \rangle$ is the duality relation in $W_2^{1/2}(S)\times W_2^{-1/2}(S)$.

Certain constructions from [14, 15] are subsequently used. It should be taken into account that, gener ally speaking, the domains of definition of the bilinear forms $\langle \mathbf{v}, \mathbf{t}^{(\nu)}(\mathbf{u})\rangle$ and $\langle \boldsymbol{\lambda}, \mathbf{t}^{(\nu)}(\mathbf{v})\rangle$ (when $\boldsymbol{\lambda} \equiv \mathbf{u}$) are different: the first of them, by virtue of the equality (which follows from Betti's formula if $\mathbf{Au} = 0$ in G, see (1.2))

$$\int_S \mathbf{v}\mathbf{t}^{(\nu)}(\mathbf{u})ds = 2\int_G W(\mathbf{u},\mathbf{v})dG$$

is symmetric and positive definite when the conditions for the unique solvability of problem (1.2) [10]

$$\int_G \mathbf{u}dG = \int_G \mathbf{rot}\,\mathbf{u}dG = 0 \qquad (2.5)$$

are satisfied and the corresponding space of the tracks $W_2^{*1/2}(S)$ forms [14] a subspace in $W_2^{1/2}(S)$. The second form is defined in the whole of the space $W_2^{*1/2}(S)$. Hence, in every case, the expression $\langle \boldsymbol{\lambda}, \mathbf{t}^{(\nu)}(\mathbf{v})\rangle$ (when $\boldsymbol{\lambda} \equiv \mathbf{u}$) has a meaning each time the expression $\langle \mathbf{v}, \mathbf{t}^{(\nu)}(\mathbf{u})\rangle$ has a meaning. Next, in the case of the first form, the following representation holds [14] (on the basis of the Riesz theorem on the general form of a linear continuous functional in Hilbert space): $\langle \mathbf{t}^{(\nu)}(\mathbf{u}), \mathbf{v}\rangle = [\mathbf{u}, \mathbf{v}]_{1/2,S} = (\mathbf{Tu}, \mathbf{v})_{O,S}\ \forall \mathbf{v} \in W_2^{*1/2}(S)$, where $[...]_{1/2,S}$, $(...)_{O,S}$ are scalar products in $W_2^{*1/2}(S)$, $L_2(S)$ respectively and $\mathbf{T}$ is a self-adjoint operator in $W_2^{*1/2}(S)$. In the case of the second form, by virtue of the generalized Schwartz inequality $|\langle \mathbf{u}, \mathbf{t}^{(\nu)}(\mathbf{v})\rangle| \leq \|\mathbf{u}\|_{1/2,S}\|\mathbf{t}^{(\nu)}(\mathbf{v})\|_{-1/2,S}$, the alternative representations

$$\langle \mathbf{u},\mathbf{t}^{(\nu)}(\mathbf{v})\rangle = (\mathbf{u},\mathbf{T}_0^{-1}\mathbf{t}^{(\nu)}(\mathbf{v}))_{1/2,S} \quad \forall \mathbf{u} \in W_2^{1/2}(S)$$

$$\langle \mathbf{u},\mathbf{t}^{(\nu)}(\mathbf{v})\rangle = (\mathbf{T}_0\mathbf{u},\mathbf{t}^{(\nu)}(\mathbf{v}))_{-1/2,S} \quad \forall \mathbf{t}^{(\nu)}(\mathbf{v}) \in W_2^{-1/2}(S)$$

hold where $\mathbf{T}_0$ is the canonical isometry of $W_2^{1/2}(S)$ in $W_2^{-1/2}(S)$ which is determined (according to the Riesz theorem mentioned above) by the relationship $(\mathbf{u}, \mathbf{v})_{1/2,S} = (\mathbf{T}_0\mathbf{u}, \mathbf{v})_{O,S} = (\mathbf{T}_0\mathbf{u}, \mathbf{T}_0\mathbf{v})_{-1/2,S}\ \forall \mathbf{u}, \mathbf{v} \in W_2^{1/2}(S)$ and is a completely continuous operator. From this, using the second of the representations when $\mathbf{t}^{(\nu)}(\mathbf{v}) \equiv \mathbf{T}_0\mathbf{v}$, we obtain $\langle \mathbf{u}, \mathbf{t}^{(\nu)}(\mathbf{v})\rangle = (\mathbf{T}_0\mathbf{u}, \mathbf{v})_{O,S}\ \forall \mathbf{v} \in W_2^{1/2}(S)$.

It remains to represent the linear continuous functional $\langle \mathbf{g}^{(\nu)}, \mathbf{v}\rangle$ (when $\mathbf{g}^{(\nu)} \in W_2^{-1/2}(S)$ in the form

$$\langle \mathbf{g}^{(\nu)}\mathbf{v}\rangle = (\mathbf{Q},\mathbf{v})_{O,S}, \quad \mathbf{Q} \in W_2^{-1/2}(S), \quad \forall \mathbf{v} \in W_2^{1/2}(S).$$

As a result, by summing the results which have been obtained, Eq. (2.2) is reduced to an operator equation. From the point of view of spectral analysis, the result obtained reduces to the use of the well-known Weyl theorem (see [16], for example) on the preservation of a limiting spectrum of a self-adjoint operator when it is perturbed by a completely continuous self-adjoint operator. Actually, the limiting spectrum of a self-adjoint bounded operator $\mathbf{T}$ ($\|\mathbf{Tu}\|_{-1/2,S} \leq c\|\mathbf{u}\|_{1/2,S}$, $c > 0$ (see [14]) is contained in a segment $[c_0, \|\mathbf{T}\|_{-1/2,S}]$, where $c_0 > 0$ is a constant from the condition of positive definiteness [14]. By virtue of the above theorem due to Weyl, the limiting spectrum of an operator $\mathbf{T} + \mathbf{T}_0$ is also contained in the indicated segment. Consequently, zero does not belong to the spectrum of this operator and, correspondingly, the matrix generated by it in the approximations $\{\mathbf{u}_k\}$ is non-singular, which it was required to prove.

As has already been noted, after determining the coefficients U_i $(i=1, \ldots, k)$, the coefficients T_j $(j=1, \ldots, n)$ (see (2.4)) are determined from the subsystem obtained from Eq. (2.3), the matrix of which is formed from coefficients of the form $\int \varphi_l \psi_j ds$ $(l=1, \ldots, k;$ $j=1, \ldots, n)$ and its non-singularity can be ensured by the choice of the approximating systems of functions (see (2.4)). Returning to the case under consideration when $S_0 \subset S$, we note that the calculations carried out for the bilinear form $\langle \boldsymbol{\lambda}, \mathbf{t}^{(\nu)}(\mathbf{v})\rangle$ (when $\lambda \equiv \mathbf{u}$) also hold in the case of its contraction on S_0. Here, account is taken of the fact that $\|\mathbf{v}\|_{1/2,S_0} \leqslant \|\mathbf{v}\|_{1/2,S}$ $\forall \mathbf{v} \in W_2^{1/2}(S)$ and the sign of the equality holds if $\mathbf{v}|_{S\setminus S_0}=0$.

The solutions of system (2.2), (2.3) can be analysed. Using (2.4) for each $i \in [1, k]$ and $j \in [1, n]$ we obtain from Eqs (2.2) and (2.3)

$$U_i = Q_i a_{il}(b_{il} + b_{il}^0)^{-1}, \quad T_j = U_i b_{il}^0 (c_{jl}^0)^{-1}$$

where the coefficients a_{il}, b_{il} $(i, l=1, \ldots, k)$ are defined in terms of integrals with over S of products of the functions φ_i, φ (and their derivatives), while the coefficients b_{il}^0, c_{jl}^0 $(i, l=1, \ldots, k;$ $j=1, \ldots, n)$ are defined in terms of integrals over S_0. The connectivity of the solutions in terms of the coefficient b_{il}^0, which is the result of the realization of the coupling equation (2.3) at the points S_0, is obvious and, here, if $b_{il}^0=0$ ($S_0=0$ in the case when there is no singular point), then the solution $\mathbf{u}_i = U_i\varphi_i$, $U_i = Q_i a_{il} b_{il}^{-1}$ $(i, l=1, \ldots, k)$ is a "Ritz" solution of a regular second problem in the theory of elasticity.

We note that approximations, similar to (2.4), have been considered previously [17, 18] in the realization of duality algorithms for solving problems involving the minimization of generalized Trefftz functionals of second-order elliptic boundary-value problems. It was established that they converge to the exact solution of the variational problem.

3. Changing to the finite-element methods approximations, let us describe the VMBE algorithm (which is similar to the algorithm described in [8]). In the case of the coupled approximation, the displacement field at the points of a boundary element (BE) is interpolated using the grid point values. This is "local interpolation". The global interpolation function at the points of a discrete boundary is constructed taking account of the condition for the matching of the boundary elements. The grid point values are equal at the common grid points of adjacent elements which corresponds to the conformal version of finite-element methods [11]. The stress field is a derivative of the displacement field. The set D (see (1.2)) is approximated by a sequence of discrete boundary potentials with a density in the form of interpolation functions, the grid point values of which are determined from a finite dimensional variational equation (from the approximating equation (1.3)) which is transformed into a system of discrete boundary equations (DBE).

Let us now consider a scheme for realizing a locally uncoupled formulation of VMBE for the numerical calculation of the singularity in a stress field in the neighbourhood of a singular point $y_0 \in S$. Let $S_\Delta = U\Delta s_n$ $(n=1, \ldots, N)$ be the discrete boundary, let Δs_n be the boundary elements and let

$$y_\Delta^{(i)} = \sum_{n=1}^{N} \sum_{k=1}^{K} Y_{nk}^{(i)} \psi_k, \quad i=1,\ldots,m \tag{3.1}$$

be the parametric equation of S_Δ, where $Y_{nk}^{(i)}$ are the Cartesian (global) coordinates of the grid points $k \in \Delta s_n$ and $\varphi_k(\eta)$ are the basis functions of the finite-element method and η is the local coordinate of the points Δs_n. When the superposition principle described above is used in the case of the vector of the specified stresses, a singular stress field (due to the effect of the singular point) is superimposed on the regular stress field and the discretization of the boundary and boundary-element approximation must also correspond to this principle: a subparametric approximation is locally superimposed (in the neighbourhood of the singular point) on an isoparametric approximation. Here, one of the rigid points of the isoparametric interpolation is made coincident with the singular point and the subparametric interpolation uses multiple grid points in the neighbourhood of the singular point. Hence, the locally uncoupled finite element approximation is written in the form

$$\mathbf{u}_N = \sum_{n=1}^{N}\sum_{k=1}^{K} \mathbf{U}_{nk}\psi_k, \quad \boldsymbol{\lambda}_N \equiv \mathbf{u}_N, \quad \mathbf{t}_N^{(\nu_N)} = \sum_{k'=1}^{K'} \mathbf{T}_{Nk'}\psi'_{k'} \tag{3.2}$$

Here, $\mathbf{u}_N$ is the isoparametric approximation of the displacement vector at the points S_Δ, where $\mathbf{U}_{nk}$ is the vector of the grid point values at the grid points $\{k\}$ of the isoparametric interpolation, the coupled finite element approximation of the vector $\mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N)$ is determined on the whole of the boundary S_Δ, $\mathbf{t}_N^{(\nu_N)}$ is the local interpolation of the stress field at the points of the "singular" finite element Δs_N (in the neighbourhood of the singular point), where $\mathbf{T}_{Nk'}$ is the vector of the grid point values of the subparametric interpolation using basis functions ψ'_k of a higher order than $\psi_{k'}$ and the set of grid points $\{k'\}$ includes the multiple gridpoints k'_-, k'_+ in the neighbourhood of the singular point. The element Δs_N therefore corresponds to the discrete part $S_{o\Delta}$. Here, a certain analogy with the special elements used in [19] for modelling a singularity in the neighbourhood of the crack tip is examined.

The finite-element approximation (4.2) [9] is an approximation of the solution of the duality problem in the case of an approximating Lagrangian $L_{o\Delta}(\mathbf{u}_N, \mathbf{t}_N^{(\nu_N)}, \boldsymbol{\lambda}_N)$ (see (2.1)) which is equivalent to a finite-dimensional problem for a boundary functional (BF) $F_\Delta(\mathbf{u}_N)$ which approximates problem (1.2). The solution of the above-mentioned problem reduces [9] to the solution of a system of discrete boundary equations

$$2\sum_{n=1}^{N}\sum_{k=1}^{K} \mathbf{U}_{nk}b_{kl}^n + \sum_{k=1}^{K} \mathbf{U}_{Nk}b_{lk}^N = 2\sum_{n=1}^{N}\sum_{k=1}^{K} \mathbf{Q}_{nk}a_{kl}^n \tag{3.3}$$

$$\sum_{k'=1}^{K'} \mathbf{T}_{Nk'}c_{k'l}^N = \sum_{k=1}^{K} \mathbf{U}_{Nk}b_{kl}^N, \quad l = 1,\ldots,K \tag{3.4}$$

where the coefficients a_{kl}^n, b_{kl}^n, c_{kl}^n are determined [8] as contributions from adjacent elements Δs_n for which a grid point k is common.

It is obvious that the structure of Eqs (3.3) and (3.4) corresponds to the structure of Eqs (2.2) and (2.3) in the approximations (2.4). The algorithm for solving Eqs (3.3) and (3.4) also corresponds to that which has been considered above: a "global" system of discrete boundary equations is solved with respect to $\{\mathbf{U}_{nk}\}_{k=1,\ldots,K_N}$ (K_N is the number of gridpoints of the isoparametric approximation of the displacement field at the points S_Δ (see (3.2)). A "local" system of discrete boundary equations is subsequently solved for $\{\mathbf{T}_{Nk'}\}_{k'=1,\ldots,K'}$ (K' is the number of grid points of the subparametric interpolation of the stressfield at the points $S_{o\Delta} \subset S_\Delta$).

The solvability of the "global" system follows from what has been said above (Section 2) when the conditions (see (2.5))

$$\int_{G_\Delta} \bar{\mathbf{u}}_N dG_\Delta = \int_{G_\Delta} \mathbf{rot}\, \bar{\mathbf{u}}_N dG_\Delta = 0$$

are satisfied where $\bar{\mathbf{u}}_N$ is the solution of the discrete variational problem in the domain G_Δ with a boundary S_Δ (of the approximating problem (1.2)), constructed using approximations of $\mathbf{u}_N$, $\mathbf{t}^{(\nu_\Delta)}(\mathbf{u}_N)$ on the boundary in accordance with the VMBE algorithm [8].

A known relationship between the number s_u of degrees of freedom of the finite elements (in terms of the components of the grid point displacements) and the order of the p polynomial is used to select the interpolation polynomial for the finite-element approximation of the displacement field: $s_u = \frac{1}{2}(p+1)(p+2)$ (written here for a two-dimensional element) which determines the complete polynomial. The possible number of degrees of freedom s_t in terms of the components of the grid point stresses must then satisfy the condition $s_t \geqslant s_u - s_0$, where s_0 is the number of degrees of freedom of the finite elements as a rigid whole.

4. A model planar problem on the stress–strain state of an elastic homogeneous isotropic medium with a crack under the action of stresses $\sigma^{(22)} \equiv -\sigma_0$, $\sigma^{(12)} \equiv -\tau_0$, which are uniformly distributed over the contour of the crack and give rise to a normal separation [cleavage] and transverse shear deformations, has been considered. A formulation of this problem is given in [19]. The numerical-analytic algorithm described above for modelling the singularity in the stress field in the neighbourhood of the crack tip (see (1.1)) was used where $\mathbf{g}_r^{(\nu)} = (-\tau_0)l^{(1)} + (-\sigma_0)l^{(2)}$ and $l^{(i)}$ are the direction cosines of the outward normal ν to the contour of the crack S. In order to solve Eq. (1.2), a linear isoparametric finite-element approximation of the displacement field at points of the crack contour was used in which a subparametric approximation was superimposed in the neighbourhood of the crack tip at the points of the "singular" element Δs_N. The following versions were used: the grid points of the linear approximation of the displacement field were made more dense; a cubic polynomial was used to approximate the displacement field in the linear element Δs_N; a locally uncoupled finite-element approximation of the form of (3.2) was used, where ψ_k are linear functions and $\psi'_{k'}$ are cubic functions. The numerical effect was evaluated: for the first two coupled approximations on the whole boundary S_Δ using an a posteriori estimate [8], the right-hand side of which can be written in a discrete form (using the notation of system (3.3))

$$I_{S_\Delta} = \sum_{n=1}^{N}\sum_{l=1}^{K}\mathbf{U}_{nl}\left(\sum_{k=1}^{K}\mathbf{U}_{nk}b_{kl}^{n} - \sum_{k=1}^{K}\mathbf{Q}_{nk}a_{kl}^{n}\right)$$

where $\mathbf{Q}_{nk} = \mathbf{Q}_{rnk} + \mathbf{T}_{nk}^{(ij)}$, $\mathbf{Q}_{rnk}$, $\mathbf{T}_{nk}^{(ij)}$ are the vectors of the grid-point values of the regular and singular components of the stresses specified on the crack contour (see (1.1)) while, in the case of the third version of a locally uncoupled approximation, I_{S_Δ} is supplemented by the term (see (3.4)) when $K=2$, $K'=2$

$$\sum_{l=1}^{K}\mathbf{U}_{Nl}\left(\sum_{k'=1}^{K'}\mathbf{T}_{Nk'}c_{k'l}^{N} - \sum_{k'=1}^{K'}\mathbf{T}_{Nk'}^{(ij)}c_{k'l}^{N}\right)$$

For a fixed number $N=12$ (on half of the crack contour), refinement of the linear approximation (three additional gridpoints were introduced) and the use of a cubic interpolation polynomial yielded roughly the equivalent numerical effect: $I_{S_\Delta} = 0.605; 0.545$, while the use of the locally uncoupled finite-element approximation yielded a significantly better result $I_{S_\Delta} \approx 0.22$.

Hence, also in the case of an isoparametric finite-element approximation, numerical modelling of a stress singularity in the neighbourhood of a singular point using a singular solution is possible but with insufficient accuracy. The accuracy depends on the order of the approximation at the additional gridpoints $\{k'\}$, that is, on the order of the basis functions $\psi'_{k'}$. However, account should be taken of the fact that an increase in the order does not always lead to an increase in the accuracy of the approximation. For instance, it has been noted ([11, p. 142]) that, if the solution of an elliptic boundary-value problem is not very smooth, the use of interpolation polynomials of degree $p>4$ to approximate it does not improve the accuracy of the approximation.

We shall also apply the numerical-analytic algorithm which has been described for modelling singularities in a solution for solving traditional problems in the theory of elasticity with singularities of the type of cavities, notches and cusps. In contact problems and problems concerning cracks (both planar and spatial), the algorithm enables one to realize a singularity in the stress field of the order of r_0^{-2} since the components of the stress tensor $\mathbf{T}^{(ij)}$ have the above-mentioned singularity and the stress field in the neighbourhood of the singular points is modelled using these components. Existing algorithms for modelling singularities in problems concerning cracks [19] realize a singularity in the stresses of the order of $r_0^{-1/2}$ and r_0^{-1}.

In connection with the comparison which has been made, we shall characterize certain complications in the implementation of the proposed algorithm. Naturally, an algorithm for solving a problem with a singularity is more complex compared with an algorithm for solving a regular problem: in the given case, both a refinement of the subdivision in the neighbourhood of a singular point into isoparametric elements, as well as the use of a locally subparametric interpolation leads to an increase in the order of the system of resolving equations. However, the use of a singular element (a Wilson element [19]) also leads to an increase in the number of degrees of freedom and, consequently, to an increase in the order of the system of equations.

A complication, associated with the ill-conditioned form of the matrix of a system, can arise in the choice of the size of the neighbourhood of the multiple grid points (see above). The recommendations in [13] were used here: the size is 0.05 of the length of an element Δs_N. The above-mentioned fact also has an effect on the accuracy with which the stress intensity factor can be determined.

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